Do Capital Asset Markets Have Stable Price Equilibria?

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Abstract

This paper analyzes the local stability of price equilibria in capital asset markets in terms of a class of deterministic dynamical models based on considerations of supply-demand balance, asset store-of-value function, and asset scarcity. Several model instances, simplified to ensure analytical tractability, are examined. In each case, price equilibria are found to be catastrophically unstable, in that the matrices characterizing their dynamics near equilibrium have eigenvalue spectra heavily weighted to positive values. This is evidence that the very special condition of local equilibrium stability is unlikely to be satisfied by realistic asset markets. This instability of price equilibria in asset markets undermines the notion of "asset value" that underlies the Efficient Market Hypothesis.

Keywords: Asset Pricing, Information and Market Efficiency
JEL: G12, G14

1. Introduction

The notion of the "value" of a capital asset is of central importance to the Efficient Market Hypothesis (EMH), and to the portfolio-management theory that relies upon the EMH. Operationally, an asset's value is taken to be the expectation value of its price distribution, about which its price time series is presumed to fluctuate. This expectation value is presumed to be fixed and stable, absent any new outside information about the asset (or "shock"); such information, when available, may cause adjustments in the asset's value, which are then reflected in its price time series (see Fama, 1965).

The essential intangibility of this notion of value, and of the process by which it is adjusted in the presence of new information, has given rise to critiques of

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the EMH (Peters, 1996; Cooper, 2008), as well as efforts to test empirically the observational predictions of the EMH. Fama (1970) gave an early, and authoritative review of such efforts, finding that "...with but a few exceptions, the efficient markets model stands up well". Fama (1970) reported that various careful tests of the EMH in its Weak form (all price history information already fully reflected in asset value), Semi-Strong form (all public information fully reflected in asset value), and Strong form (all information fully reflected in asset value) turn up at best weak evidence for trouble with the EMH, and for the most part no evidence at all.

Despite some empirical work (e.g. Basu, 1977; Shiller, 1981; Rosenberg et al., 1985) appearing to dispute aspects of the EMH, these conclusions largely stand today, at least insofar as the EMH is still considered solid enough to furnish the basis for standard portfolio theory (Markowitz, 1952; Sharpe, 1964; Amenc and Le Sourd, 2003). Indeed, to the extent that disputes exist, they appear to be centered more on what is to replace the clearly inadequate normal probability distribution that follows from the random-walk assumption (Mandelbrot, 1963; Fama, 1965; Peters, 1996). As Fama (1970) points out, fat tails, which can be accommodated by introducing Stable Pareto distributions, constitute evidence against random walks, but not against sub-Martingale models, which are perfectly consistent with the EMH's assumptions about the incorporation of "new information" into expectation values of price distributions. Further disputes centering about evidence that asset price time series exhibit "memory", manifesting itself as Hurst Index values in the "persistent" range $H > 0.5$ (Peters, 1996; Costa and Vasconcelos, 2003) are intriguing, but disputed on grounds of statistical significance (Couillard and Davison, 2005).

It may appear, then, that for the most part all is well with the EMH, at least from the point of view of its having largely withstood determined efforts at empirical falsification. There is, however, another source of potential trouble for the EMH, deriving from the notion of asset "value". To see the source of the problem, it is helpful to distinguish two different — and logically separate — roles that the notion of "value" plays within the EMH:

- Asset "Value" is the expectation value of an asset's price time series in the absence of new and relevant market information, as outlined above;

- Asset "Value" is the asset's *Equilibrium Price* in a competitive market, in which the various requirements of buyers and sellers, manifested as demand and supply curves, somehow come into balance, and stay there.

The various empirical tests of the EMH test the first of these two meanings of asset value. The second meaning — asset value as equilibrium price — is, quite generally, simply assumed. The classic arguments for existence and accessibility of price equilibria in markets for goods and services (see Samuelson, 1947, for example) are assumed to carry over into asset markets, so that one may simply import the idea of an equilibrium price, and use it as a proxy for the (complicated, poorly-understood) price-adjustment dynamics of the market.

This assumption warrants further examination. In the study of competitive markets for goods and services, considerable amount of work has gone into establishing not only the existence of price equilibria (Debreu, 1982; Keisler, 1996) but also the *stability* of such equilibria (Samuelson, 1941, 1947; Hahn, 1982). This last is a crucial point. Even where there are good and sufficient grounds to believe that a price equilibrium exists, in the sense of a price at which supply meets demand, that equilibrium is only useful as a proxy for the market dynamics *if it is stable*. If it isn't, then in general the solution of the dynamical system governing price will *avoid* the equilibrium point, rather than converge towards it, for general initial conditions outside a set of measure zero.

For most commodity market models, conditions for stability have been established, and appear to be reassuringly general (Hahn, 1982). Moreover, stable price equilibria in commodity markets are a matter of everyday observation and experience. It is perhaps unsurprising, then, that researchers working on asset pricing models have helped themselves to the idea of the equilibrium price as the underlying meaning of the term "value" in the context of their work. By so doing, they have circumvented the necessity of understanding the complex dynamics governing asset prices, and cut straight to the chase of modeling the empirical behavior of those prices.

But there is reason to question the wisdom of this appropriation of ideas from the theory of commodity markets. Capital assets differ substantially from goods and services in their economic properties. In the first place, capital assets are *scarce*. Their "production" is not related to "factors of production" in the standard way that is assumed in normal commodity market models, and their supply at a given price is not determined by considerations of marginal production costs.

Moreover, on the demand side, it is clear that buyers' incentives are quite different with respect to capital assets than they are with respect to commodities — simply put, we don't buy stock for the same reasons we buy potatoes, or haircuts. Capital assets have a *store of value* function that is non-existent for goods and services. That is to say, assets are partly valued for their ability to appreciate, or at least to not decline in price (relative to other assets).

Cooper (2008, p. 8) appositely writes "Whenever we invest in the hope of achieving capital gains we are seeking scarcity value, in defiance of the core principle that supply can move in response to demand...in asset markets it is the rate of change of prices that stimulates shifting demand."

This observation amounts to the introduction of new structure in the demand function for assets which profoundly affects the nature of equilibrium between their supply and demand. Demand for an asset is a function not only of its price P, but *also of the time derivative, $\dot{P}$, as well as of the prices and price rates-of-change of all the other competing assets in the market*. It should be clear that since, in the absence of production, buyers are also sellers, the supply function for capital assets sports a complementary dependence on price and price rate-of-change.

The upshot of all this is that the balance of supply and demand for assets is not an algebraic condition (leading to an equilibrium price); it is rather a *dynamical condition* — a differential equation governing the evolution of prices with time, which replaces the traditional (but wholly empirical) price modeling schemes based on stochastic processes such as random walks or sub-martingales. This dynamical system induced by the balance of supply and demand may be analyzed to infer the stability properties of any price equilibria to which it may give rise. That analysis is the central purpose of this work.

To be clear: this paper makes no empirical predictions of price distribution, based on the class of dynamical models considered. The core result here is a critique of the concept of "equilibrium price" in asset markets, and of the use made of that concept by the EMH.

The plan of the paper is as follows: §2 discusses the purported dynamics induced by supply-demand balance, how they may arise from a more general dynamical system, the nature of "equilibrium price", and certain fairly general properties of the system that are useful in its stability analysis. §3 gives examples of model market structures, of various degrees of realism (mostly toy), for which stability analysis may be performed analytically. Finally, §4 summarizes what has been learned from the analysis.

2. Dynamics, Equilibrium Dynamics, and Static Equilibrium

2.1. Equilibrium Dynamics

In what follows, a market for N assets will be assumed, where $P_i (i = 1, \ldots, N)$ is the price of the i-th asset. The P_i will be regarded as the components of an N-dimensional vector $\mathbf{P}$.

The supply function for the i-the asset will be denoted by S_i, and the demand function by D_i. These are also regarded as the i-th components of vectors $\mathbf{S}$ and $\mathbf{D}$, respectively. The excess supply function we write $B_i \equiv S_i - D_i$, i-th component of a vector $\mathbf{B} = \mathbf{S} - \mathbf{D}$.

Discussion of numeraires, money, interest rates and the like is avoided here, since these complicate the argument needlessly without really adding any substantially new feature. It may be taken as read that assets are priced in terms of some price unit — possibly in terms of the price P_i of a member of the collection of assets, possibly not.

As discussed in the introduction, $\mathbf{S}$, $\mathbf{D}$, and $\mathbf{B}$ are to be regarded as functions of price $\mathbf{P}$ *and* of price rate-of-change $\dot{\mathbf{P}}$, $\mathbf{B} = \mathbf{B}(\mathbf{P}, \dot{\mathbf{P}})$, and so on. As implied by the notation, the supply and demand for each asset is a function not only of its own price and price rate-of-change, but on those of all the other assets as well.

Note that this dependence of $\mathbf{S}$, $\mathbf{D}$, and $\mathbf{B}$ on price behavior does not also preclude additional dependence on other information, such as earnings, prevailing interest rates, splits, etc. These dependencies may be encoded as externally controllable parameters θ expressing such supplemental information — that is, $\mathbf{S} = \mathbf{S}(\mathbf{P}, \dot{\mathbf{P}}; \theta)$, and so on. However, when the factors thus parametrized are regarded as constant (i.e. no new information "shock"), they are inessential to the analysis of price equilibrium stability, so we will drop the notational dependence on θ in what follows.

It is clear then that the requirement that supply and demand should be in balance,

$$\mathbf{B}(\mathbf{P}, \dot{\mathbf{P}}) = \mathbf{S}(\mathbf{P}, \dot{\mathbf{P}}) - \mathbf{D}(\mathbf{P}, \dot{\mathbf{P}}) = 0, \tag{1}$$

is a *first-order differential equation*, rather than the algebraic condition to which we are accustomed from the case when supply and demand have no dependence on price rate-of-change. Balance between supply and demand is no longer a condition that yields price equilibrium. Instead, it yields what may be termed *Equilibrium Dynamics*.

Eq. (1) governs the evolution of asset prices $\mathbf{P}$, at least approximately, in the case when market processes occurring out of supply-demand balance produce price adjustments towards balance that are more rapid than the timescales characteristic of Eq. (1). Out-of-balance dynamics will be considered further in §2.3.

2.2. *Equilibrium and Stability*

The condition for price equilibrium is, of course, $\dot{\mathbf{P}} = 0$. Substituted into Eq. (1), we see that a price vector $\mathbf{P}_0$ is an equilibrium point of the dynamical system if it is a solution of the algebraic equation

$$\mathbf{B}(\mathbf{P}_0, \mathbf{0}) = \mathbf{0}. \tag{2}$$

Establishing the existence of such equilibria is a topic outside the scope of this work (see e.g. Debreu, 1982; Keisler, 1996). I will simply assume their existence here, in order to move on to the question of their (local) stability.

Local stability is addressed using standard methods (Samuelson, 1941, 1947; Hahn, 1982). In the first place, we linearize Eq. (1) about the equilibrium point $\mathbf{P}_0$:

$$\left(\frac{\partial \mathbf{B}}{\partial \mathbf{P}}\right)(\mathbf{P} - \mathbf{P}_0) + \left(\frac{\partial \mathbf{B}}{\partial \dot{\mathbf{P}}}\right)\dot{\mathbf{P}} = 0, \tag{3}$$

where $(\partial \mathbf{B}/\partial \mathbf{P})$ is an $N \times N$ matrix whose i, j component is $(\partial B_i/\partial P_j)_{\mathbf{P}=\mathbf{P}_0, \dot{\mathbf{P}}=\mathbf{0}}$, and similarly for $(\partial \mathbf{B}/\partial \dot{\mathbf{P}})$. The linearized dynamical equation, prepped for stability analysis, is

$$\dot{\mathbf{P}} = \left(-\frac{\partial \mathbf{B}}{\partial \dot{\mathbf{P}}}\right)^{-1}\left(\frac{\partial \mathbf{B}}{\partial \mathbf{P}}\right)(\mathbf{P} - \mathbf{P}_0). \tag{4}$$

Local stability of Eq. (4) hinges upon the eigenvalue structure of the matrix product on the RHS of the equation. Eq. (4) is locally stable if all the eigenvalues have real parts that are negative, or at worst zero.

In order to analyze the required eigenvalue spectrum, it is necessary to say more about the matrix elements of the matrices in the product. Certain properties of these matrix elements may be inferred from the effect of basic investor incentives upon the supply and demand functions, $\mathbf{S}(\mathbf{P}, \dot{\mathbf{P}})$ and $\mathbf{D}(\mathbf{P}, \dot{\mathbf{P}})$. The partial derivatives of these functions may be assumed to obey the following inequalities:

$$\frac{\partial S_k}{\partial P_k} > 0 \quad ; \quad \frac{\partial D_k}{\partial P_k} < 0 \quad ; \tag{5}$$

$$\frac{\partial S_k}{\partial \dot{P}_k} < 0 \quad ; \quad \frac{\partial D_k}{\partial \dot{P}_k} > 0 \quad ; \tag{6}$$

$$\frac{\partial S_k}{\partial P_l} < 0 \quad ; \quad \frac{\partial D_k}{\partial P_l} > 0 \quad , \quad (k \neq l); \tag{7}$$

$$\frac{\partial S_k}{\partial \dot{P}_l} > 0 \quad ; \quad \frac{\partial D_k}{\partial \dot{P}_l} < 0 \quad , \quad (k \neq l). \tag{8}$$

Eqs. (5) are the classic normal dependence of supply and demand for goods and services on price, while Eqs. (6) express the fact that a depreciating asset is a better candidate for its owner to unload than an appreciating one, while an appreciating asset is more attractive to potential buyers than a depreciating one.

Eqs. (7) express the assumption that assets are gross substitutes. They purport that the supply of the k-th asset is lower at a higher price of the l-th asset, since the higher l-price decreases the temptation to liquidate a k-position in order to acquire an l-position; and that the demand for asset k is higher at a higher price of asset l, as asset k then appears to be a better bargain by comparison with asset l.

Eqs.(8) express the extension of gross substitutability to price rate-of-change dependencies. They state that the supply of the k-th asset is higher at a higher appreciation rates of the l-th asset, since the higher l-appreciation rate makes the l-th asset more attractive, increasing the temptation to liquidate k-positions; and that the demand for asset k is lower at a higher appreciation rate of the l-th asset, since the l-th asset is then relatively more attractive than the k-th asset.

In terms of the excess supply function $\mathbf{B}$, these inequalities imply

$$\frac{\partial B_k}{\partial P_k} > 0 \quad ; \quad \frac{\partial B_k}{\partial \dot{P}_k} < 0, \tag{9}$$

$$\frac{\partial B_k}{\partial P_l} < 0 \quad ; \quad \frac{\partial B_k}{\partial \dot{P}_l} > 0, \quad k \neq l. \tag{10}$$

which means that $-(\partial \mathbf{B}/\partial \mathbf{P})$ and $(\partial \mathbf{B}/\partial \dot{\mathbf{P}})$ are so-called Metzler matrices.

As we will see in §3, inequalities (9) and (10) are fundamental to addressing the question of local stability of Eq. (4), in the context of models of the local functional dependence of $\mathbf{B}$ on $\mathbf{P}$ and $\dot{\mathbf{P}}$.

2.3. *Out-Of-Balance Dynamics*

It is possible to feel some discomfort with the idea of using Eq. (1) as the dynamical system governing the evolution of asset prices. After all, Eq. (1) presumes that supply and demand are always in balance. Perturbations such as Eq. (3) only perturb the equilibrium price along directions consistent with such balance. In general, we should expect this in-balance dynamical system to be embedded in a more general dynamical system, one that allows for the possibility of unrequited buy or sell orders, at least on short timescales. Such a generalized dynamical system may be of some relevance to the perturbation analysis, even if the equilibrium configuration about which the perturbation is performed is in fact on the surface $\mathbf{B} = \mathbf{0}$, because the extra direction out of the surface could, in principle, have a stabilizing — or destabilizing — effect on the eigenvalue spectrum of the linearized dynamical system.

In order to consider this possibility, it is necessary to assume a reasonable dynamical system that reduces under balance conditions, to Eq. (1). The standard procedure (Hahn, 1982) is to assume a first-order dynamical equation,

$$\dot{\mathbf{P}} = \mathbf{H}(\mathbf{B}), \tag{11}$$

where the action lives in the choice of the driving term $\mathbf{H}(\mathbf{B})$, which generally features the property that $\mathbf{H}(\mathbf{0}) = \mathbf{0}$. Tatonnement models frequently set $\mathbf{H} = -\lambda\mathbf{B}$ ($\lambda > 0$), so that if an asset has non-zero excess supply, its price moves in such a way as to abate the excess.

If this strategy were adopted, the apparently novel feature would be the appearance of a dependence on price rate-of-change $\dot{\mathbf{P}}$ in $\mathbf{B}$. Actually, this would not be such a novelty: for example, the Tatonnement models of Enthoven and Arrow (1956) and of Arrow and Nerlove (1958) already feature dependence on price rate-of-change in the driving term, in order to provide a movement towards expected (extrapolated) future value.

However, in the present context, a dynamical equation such as Eq. (11) is too restrictive. It enforces $\mathbf{B} = \mathbf{0} \Rightarrow \dot{\mathbf{P}} = \mathbf{0}$ — that is supply-demand balance necessarily entails equilibrium. That's too strong an assumption here, since we expect that there is scope for dynamic price behavior even in balance. So instead of Eq. (11), let us model out-of-balance dynamics by

$$\frac{d}{dt}\mathbf{B}(\mathbf{P}, \dot{\mathbf{P}}) = \mathbf{H}(\mathbf{P}, \dot{\mathbf{P}}), \tag{12}$$

which we will assume reduces to

$$\frac{d}{dt}\mathbf{B}(\mathbf{P}, \dot{\mathbf{P}}) = -\lambda\mathbf{B}(\mathbf{P}, \dot{\mathbf{P}}), \tag{13}$$

approximately, near equilibrium.

Eqs. (12) and (13) are *second-order* differential equations in time for $\mathbf{P}$. Equilibrium, in this generalized context, means $\mathbf{P} = \mathbf{P}_0, \dot{\mathbf{P}} = \mathbf{0}, \ddot{\mathbf{P}} = \mathbf{0}$, so that balance ($\mathbf{B} = 0$) is a necessary condition for equilibrium, although no longer a sufficient condition.

The constant λ regulates the rate at which supply-demand balance is restored in Eq. (13). We see intuitively that a stability analysis based on the in-balance dynamics of Eq. (1) is justified so long as the characteristic timescales that arise from the spectrum of Eq. (4) are long compared to $1/\lambda$.

When this condition is not fulfilled, it is necessary to perturb Eq. (13), instead of Eq. (1). This is easily done:

$$\left(\frac{\partial \mathbf{B}}{\partial \mathbf{P}}\right)\dot{\mathbf{P}}+\left(\frac{\partial \mathbf{B}}{\partial \dot{\mathbf{P}}}\right)\ddot{\mathbf{P}}=-\lambda\left[\left(\frac{\partial \mathbf{B}}{\partial \mathbf{P}}\right)(\mathbf{P}-\mathbf{P}_0)+\left(\frac{\partial \mathbf{B}}{\partial \dot{\mathbf{P}}}\right)\dot{\mathbf{P}}\right]. \tag{14}$$

We may cast this as a first-order ODE in a larger vector space:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{P}-\mathbf{P}_0\\ \dot{\mathbf{P}}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ \lambda\mathbf{G} & \mathbf{G}-\lambda\mathbf{1}\end{pmatrix}\begin{pmatrix}\mathbf{P}-\mathbf{P}_0\\ \dot{\mathbf{P}}\end{pmatrix}, \tag{15}$$

where we have defined

$$\mathbf{G}\equiv\left(-\frac{\partial \mathbf{B}}{\partial \dot{\mathbf{P}}}\right)^{-1}\left(\frac{\partial \mathbf{B}}{\partial \mathbf{P}}\right), \tag{16}$$

which is the matrix that appears on the RHS of Eq. (4).

We may express an eigenvector of the matrix on the RHS of Eq.(15) in partitioned form, that is

$$\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ \lambda\mathbf{G} & \mathbf{G}-\lambda\mathbf{1}\end{pmatrix}\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}=\mu\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}. \tag{17}$$

after some minor algebra, we find that $\mathbf{v}_2=\mu\mathbf{v}_1$, and

$$(\mu+\lambda)\mathbf{G}\mathbf{v}_1=\mu(\mu+\lambda)\mathbf{v}_1. \tag{18}$$

That is to say, if $\mathbf{v}_1$ is an eigenvector of $\mathbf{G}$ — that is, of the RHS of the equilibrium-dynamical problem of Eq. (4) — then the vector $\mathbf{v}^T=(\mathbf{v}_1^T,\mu\mathbf{v}_1^T)$ is an eigenvector of the RHS of Eq. (15). If $\mathbf{G}\mathbf{v}_1=\omega\mathbf{v}_1$, then either $\mu=-\lambda$ or $\mu=\omega$.

What we learn from all this is that even in the case when the "restoring force" that enforces supply-demand balance acts slowly compared to the dynamical times associated with Eq. (4) (that is, roughly speaking, when $\lambda<\omega$) we may proceed by analyzing the stability problem associated with Eq. (4), since it is straightforwardly associated with the more general stability problem. In particular, the general dynamical system is evidently stable if, and only if, the equilibrium-dynamical system is stable.

Consequently, in the examples discussed in §3 below, the discussion is restricted to the stability issues attending Eq. (4).

3. Some Illustrative Examples

In order exhibit analytical results on the local stability properties of Eq. (4), it is instructive to build some simple, tractable models incorporating the constraints of Eqs. (9) and (10). This section presents a selection of such models, together with their attendant stability analyses.

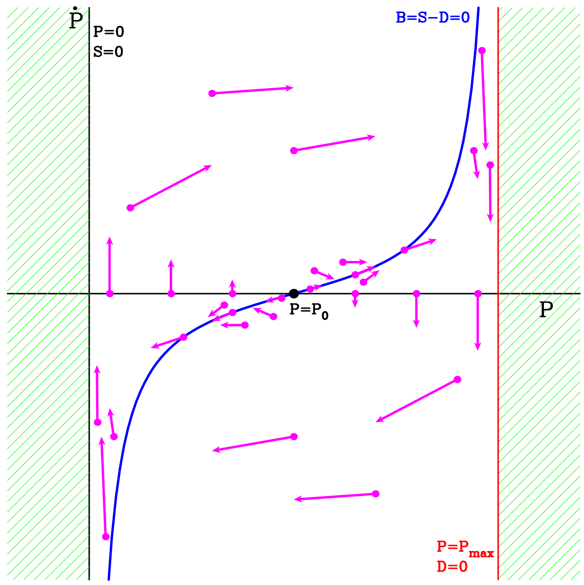

Figure 1: Phase flow and supply-demand balance for one-asset dynamics.

3.1. One-Asset Market

The simplest possible case, and the one most obviously to be examined first, is that of a market in which a single asset is traded. Restricted to $N = 1$, Eq. (4) now reads

$$\dot{P} = -\frac{\partial B/\partial P}{\partial B/\partial \dot{P}} \times (P - P_0), \tag{19}$$

where the derivatives are evaluated at $P = P_0, \dot{P} = 0$. The elementary solution is

$$P(t) - P_0 = A \exp(\omega t), \tag{20}$$

where A is a constant which may be either positive or negative, depending on initial conditions, and $\omega \equiv -(\partial B/\partial P)/(\partial B/\partial \dot{P}) > 0$, in virtue of Eqs. (9).

It is clear from Eq. (20) that equilibrium points of single-asset markets are *unstable*. The price P is repelled from the equilibrium at a locally-exponential rate ω. For general initial conditions, the equilibrium is not attainable, and if we should happen to artificially start the system with $P(t = 0) = P_0$, any slight perturbation (such as the “shock” of new information flowing to investors) will cause the market to leave equilibrium at a locally exponential rate.

Of course, P doesn't really go negative, or to infinity, as the local behavior expressed by Eq. (20) appears to imply. The price cannot go to zero in supply-demand balance, because the supply function S is zero there, and D cannot follow it. Similarly, the price cannot go to infinity in supply-demand balance, because investor budget constraints set some maximum price P_{max}, above which demand D goes to zero, at which point it necessarily is parted from S.

Thus what must happen, once the price movement of Eq. (20) has progressed sufficiently, is that the supply-demand balance breaks down in order to protect the barriers at $P = 0$ and at $P = P_{max}$. These barriers are necessarily encoded in the function $H(P,\dot{P})$ in Eq. (12). It is of some interest to consider how the dynamics of Eq. (12) cause a tearing of the balance in the one-asset case (despite the fact that this constitutes a bit of a departure from the equilibrium-stability core results of this work) because the one-asset case offers the possibility of visualizing the dynamical situation in a way that is more difficult in higher-dimensional cases.

We need to consider the situation in the full dynamical context of Eq. (12). This is depicted schematically in Fig. 1. The figure shows the phase space, with P along the horizontal axis and $\dot{P}$ along the vertical axis. The $P = 0$ barrier (where $S = 0$) is the black line at the left of the figure, while the $P = P_{max}$ barrier (where $D = 0$) is the red line at the right. The green hatched regions are forbidden to the flow.

The blue line is $B = S - D = 0$, the line of supply-demand balance. It meets the horizontal axis at the equilibrium point $P = P_0$, $\dot{P} = 0$. It necessarily asymptotes to $\dot{P} \to -\infty$ at $P = 0$ (since D cannot be zero for $P = 0$ and finite $\dot{P}$) and to $\dot{P} \to +\infty$ at $P = P_{max}$ (since S cannot be zero for $P = P_{max}$ and finite $\dot{P}$).

The magenta arrows depict the flow vector field embodied by $H(P,\dot{P})$ in Eq. (12). The vectors must point to the right above the horizontal axis, where $\dot{P} > 0$, and to the left below the axis, where $\dot{P} < 0$. On the $\dot{P} = 0$ axis, the vectors must be vertical. At the equilibrium point, the vector field must be zero.

At the extreme right of the $\dot{P} > 0$ half-plane, the flow must be strongly downward, representing the strong price deceleration required to keep P from crashing through the $P = P_{max}$ budget barrier. The closer P is to P_{max}, the larger the deceleration vector. Similarly, at the extreme left of the $\dot{P} < 0$ half-plane, the flow must be strongly upwards, to prevent the price from crashing through the zero bound.

In the quiet little pond near the equilibrium point, the flow is as implied by Eqs. (13) and (19): a restoring force leading back to the line of supply-demand balance, and a flow along the line leading away from the equilibrium point. However, as the solution moves away from equilibrium along the balance line, it begins to sense the effect of the vector field component whose job it is to protect the budget and zero-bound barriers. Eventually the barrier protection effect, which is small near equilibrium, starts to dominate the flow, and causes the tearing away of supply from demand. The trajectory in phase space then cycles around, executing orbits about the equilibrium point that are, in general, not closed curves in phase space.

We see, then, that the necessity of imposing budget and zero-bound constraints as global properties of the one-asset dynamical system can produce non-trivial dynamical behavior in the phase flow, capable of breaking down the balance of supply and demand in order to protect the constraints, and resulting in complex — and coherent — price motion without any change in the position of the equilibrium price.

These considerations certainly apply with equal force to multiple-asset models, such as the ones considered in the following sub-sections. In fact, one would expect the structure of the phase flows to be richer, as the flows can twist around in higher-dimensional spaces. However, it is also more complicated to characterize the global properties of these flows than it is for the single asset case. We therefore do not attempt this sort of analysis in the sub-sections that follow.

3.2. Uniform Prices of Asset Appreciation Rates

Another model in which Eq. (4) is easily diagonalized may be constructed by setting

$$\left(\frac{\partial \mathbf{B}}{\partial \dot{\mathbf{P}}}\right) = -\left(\frac{\partial \mathbf{B}}{\partial \mathbf{P}}\right)\mathbf{D}, \tag{21}$$

where $\mathbf{D}$ is a diagonal matrix, $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_N)$, with $d_i > 0, i = 1, \ldots, N$. Note that the positivity of the d_i is required to ensure that the model respects the inequalities of Eqs. (9) and (10).

In this model, we have that

$$\partial B_i / \partial \dot{P}_k = -d_k \partial B_i / \partial P_k. \tag{22}$$

This may be interpreted as saying that near equilibrium, it is a matter of aggregate indifference to investors whether asset k increases in price by an amount ΔP_k or loses an amount $\Delta \dot{P}_k \equiv \Delta P_k / d_k$ from its appreciation rate $\dot{P}_k$. In other words, d_k may be viewed in this model as the market price ascribed to changes in the appreciation rate of asset k. In this model appreciation rates have "uniform" prices in the sense that in Eq. (22), the value of the appreciation rate of asset k is the same measured relative to the excess supply of all assets $i = 1, \ldots, N$.

Plugging Eq. (21) into Eq. (4) we immediately obtain

$$\dot{\mathbf{P}} = \mathbf{D}^{-1}(\mathbf{P} - \mathbf{P}_0). \tag{23}$$

Since the d_i are necessarily positive, it follows that in this model we find catastrophic instability — there is not a single stable direction in the problem.

3.3. Fungible Asset Markets

Here we consider a model in which

$$\frac{\partial B_i}{\partial P_k} = (c_1 + c_2)\delta_{ik} - c_2; \tag{24}$$

$$\frac{\partial B_i}{\partial \dot{P}_k} = -(c_3 + c_4)\delta_{ik} + c_4, \tag{25}$$

with $c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0$, so that the inequalities of Eqs. (9) and (10) are satisfied. Such a market is "fungible," in the sense that there is nothing to choose between different assets — asset labels may be interchanged with no effect on the dynamics.

It is straightforward to show that the matrices of Eqs.(24) and (25) commute with each other, so that they they have common eigenvectors and may be diagonalized simultaneously. Furthermore, it is easily verified that any vector $\mathbf{x}$ whose components sum to zero (so that $\mathbf{u}^T\mathbf{x} = 0$, where $\mathbf{u}$ is a vector whose components satisfy $u_i = 1, i = 1, \ldots, N$) is an eigenvector of both matrices:

$$\left(\frac{\partial \mathbf{B}}{\partial \mathbf{P}}\right)\mathbf{x} = (c_1 + c_2)\mathbf{x}; \tag{26}$$

$$\left(\frac{\partial \mathbf{B}}{\partial \dot{\mathbf{P}}}\right)\mathbf{x} = -(c_3 + c_4)\mathbf{x}. \tag{27}$$

The subspace of such vectors is $(N-1)$-dimensional. It follows that the matrix product on the RHS of Eq. (4) has at least an $(N-1)$-dimensional degenerate irreducible subspace characterized by a positive-definite eigenvalue $\omega = (c_1 + c_2)/(c_3 + c_4)$. The remaining eigenvalue — corresponding to the eigenvector $\mathbf{x} = \mathbf{u}$ — may be positive or negative, depending on the relative sizes of the c_i. This model therefore also features highly unstable price equilibria.

3.4. Fungible Cross-Responses

Here we consider a model where the excess supply $\mathbf{B}$ responds to price changes as follows:

$$\left(\frac{\partial \mathbf{B}}{\partial \mathbf{P}}\right) = \mathbf{D}_1 - \mathbf{a}_1 \mathbf{u}^T; \tag{28}$$

$$\left(\frac{\partial \mathbf{B}}{\partial \dot{\mathbf{P}}}\right) = -\mathbf{D}_2 + \mathbf{a}_2 \mathbf{u}^T, \tag{29}$$

where as in §3.3, $\mathbf{u}$ is the vector whose components are all equal to 1. The vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are assumed to have strictly positive components $a_{si}, i = 1, \ldots, N, s = 1, 2$; and $\mathbf{D}_s \equiv \mathrm{diag}(\mu_{s1} + a_{s1}, \mu_{s2} + a_{s2}, \ldots, \mu_{sN} + a_{sN})$, $s = 1, 2$. To satisfy the inequalities of Eqs. (9) and (10), we must require that all the $\mu_{si} > 0, i = 1, \ldots, N, s = 1, 2$. As a further simplification, we assume that $\mathbf{D}_2 = \alpha \mathbf{D}_1$, with $\alpha > 0$ so as to still satisfy the constraints.

The "story" that accompanies this model is that the aggregate excess supply corresponding to each asset has its own personalized response to changes in its own price or to its own price rate-of-change (the μ parameters). However, the response of each asset's excess supply to changes in prices and price rates-of-change of other assets is uniform, so that relative to each asset the values of all the other assets are fungible. The parameter α is similar to the market price of asset appreciation, the d_i parameters in §3.2. However, here its definition is less easily interpreted as a market price of appreciation, since it is a constant of proportionality between parameter combinations $\mu_{1k} + a_{1k}$ and $\mu_{2k} + a_{2k}$.

The required inverse of the matrix in Eq. (29) may be obtained by the Sherman-Morrison-Woodbury formula (Golub and Van Loan, 1989, p. 51):

$$\left(-\frac{\partial \mathbf{B}}{\partial \dot{\mathbf{P}}}\right)^{-1} = {\mathbf{D}_2}^{-1} + \frac{1}{1-c} {\mathbf{D}_2}^{-1} \mathbf{a}_2 \mathbf{u}^T {\mathbf{D}_2}^{-1}, \tag{30}$$

where $c \equiv \mathbf{u}^T {\mathbf{D}_2}^{-1} \mathbf{a}_2 = \sum_{i=1}^{N} a_{2i}/(\mu_{2i} + a_{2i})$.

Plugging Eqs. (30) and (28) into Eq. (4), we obtain

$$\dot{\mathbf{P}} = \left\{ \alpha^{-1} \mathbf{1} - {\mathbf{D}_2}^{-1} \left[\mathbf{a}_1 + \frac{f - \alpha^{-1}}{1-c} \mathbf{a}_2 \right] \mathbf{u}^T \right\} (\mathbf{P} - \mathbf{P}_0), \tag{31}$$

where $f \equiv \mathbf{u}^T {\mathbf{D}_2}^{-1} \mathbf{a}_1 = \sum_{i=1}^{N} a_{1i}/(\mu_{2i} + a_{2i})$.

By inspection, we see immediately that just as with the model of §3.3, any vector $\mathbf{x}$ with components that sum to zero (so that $\mathbf{u}^T \mathbf{x} = 0$) is an eigenvector of the matrix in braces on the RHS of Eq. (31) with positive eigenvalue α^{-1}. That is to say, there is again an $(N-1)$-dimensional irreducible degenerate subspace with positive eigenvalue α^{-1}. The remaining eigenvector $\mathbf{D}_2{}^{-1}[\mathbf{a}_1+\mathbf{a}_2(f-\alpha^{-1})/(1-c)]$ has an eigenvalue which may be positive or negative, depending on the parameters. It is therefore clear that the equilibrium point $\mathbf{P}_0$ is again highly unstable.

4. Conclusions

To review briefly: the traditional, wholly empirical, stochastic-process-based models of capital asset pricing have been replaced here with a class of deterministic dynamical models, whose general structure is inferred by taking seriously (1) the store-of-value function of assets, (2) the scarcity of assets, and (3) the balance of supply and demand for assets. This has been done with a view to analyzing the stability of the equilibrium points of asset price dynamics. Equilibrium stability is a necessary validity condition for the adoption of equilibrium price as the notion of "asset value," because in the absence of stability, solutions of the dynamical equations governing asset prices are repelled from equilibrium points, rather than attracted towards them. But equilibrium price may only be pressed into service as a proxy for the full dynamical system if the system's solutions spend most of their time at or near the equilibria. Thus the point of the exercise is to determine whether the notion of "asset value" as an equilibrium price is rigorously justifiable, or even plausible.

The equilibrium of a set of analytically tractable toy models has been studied. In each case, the models not only do not give rise to stable equilibria: their equilibrium points are typically catastrophically unstable, with either every or almost every local mode corresponding to a positive eigenvalue.

This in no way constitutes "proof" that all asset markets lack equilibrium stability, of course. On present evidence it cannot be excluded that a model could be proposed that is consistent with the constraints on excess supply expressed by Eqs. (9) and (10) and yet contrives to keep all its eigenvalues negative, so as to ensure local stability. It should be noted, however, that to the extent that the models examined above are oversimplified and contrived, they are contrived to achieve analytical tractability, and *not* to prejudge the issue of local stability. It just turns out that all the simple models that spring to mind are not just locally unstable, but strongly unstable — their spectra appear strongly biased towards positive eigenvalues. From the perspective of this experience, it would seem that a high degree of unrealistic contrivance would be required to produce a *stable* model of asset price dynamics.

On reflection, this is not a terribly surprising conclusion. Local stability is a very special condition, which requires considerable effort to arrange and to verify even in classical markets without excess-supply dependencies on price rate-of-change (Hahn, 1982). Despite the obvious attractiveness of the notion that one can simply use an equilibrium price as a proxy for complicated and poorly-understood dynamical laws, it was always risky to assume that one could summarily take equilibrium stability as read in asset markets, without attempting to verify the plausibility of this assumption.

If we accept these conclusions, we must face up to the fact that "asset value" is a metaphysically empty concept. There is no value, only price. The operational definition of value — the expectation value of a price time series in the absence of new and relevant information — is still available, but it is not possible to attach it to the conception of equilibrium price in a competitive market model. Whatever empirical regularities those expectation values may exhibit, they must be explained in terms of the market zooming around price space under the power of its own dynamics, as much as reacting to the shocks of new information. As we saw in §3.1 in the case of a single-asset market, the dynamical system is perfectly capable of executing complex, non-stochastic motions in price space without requiring any external jiggling of the equilibrium price P_0.

In this light, the EMH's view of value as a hidden variable, incorporating all or most market information, changing only in response to changes in such information, and imposing observational consequences on price time series, acquires a somewhat theological tinge. One may as well speak of an asset's soul as of its "value".

None of this is to deny the worth of empirical studies of the statistical properties of asset markets, such as those conducted under the aegis of the EMH (or in attempts to refute it) (e.g Fama, 1970; Mandelbrot, 1963; Shiller, 1981; Costa and Vasconcelos, 2003, and many others). But it is important that one should have a clear idea of what is — and isn't — being measured and modeled. The function of ideas such as "value-as-equilibrium-price" is at least in part paradigmatic, in that it helps guide what sort of experiments, observations, and analyses we do, and which we don't bother with. If an idea as fundamental to our thinking as the meaning "asset value" is incorrect, this probably means that many research programs in this area are ripe for re-thinking.

Acknowledgements

The author is grateful to Alfred Kanzler, for useful discussion and for contributing valuable ideas to this research.

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